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CALCULATION OF THE COMPLEX ZEROS OF THE FUNCTION $P(z)$ COMPLEMENTARY TO THE INCOMPLETE GAMMA FUNCTION.*

By Philip Franklin.

The gamma function of the complex variable may be written as the sum of an integral function and the function

$$P(z) = \sum_{\nu=0}^{\infty} \frac{(-1)^\nu}{\nu!} \frac{1}{z+\nu}$$

which has the same poles and the same residues as $\Gamma(z)$ itself. The fact that $P(z) = 0$ has exactly four complex roots was first stated, although with an insufficient proof, by L. Bourguet.† The method of computing these four complex roots used here is based on an existence proof given recently by T. H. Gronwall.‡ This proof consists in first showing that the auxiliary rational function

$$P_n(z) = \sum_{\nu=0}^{2n+1} \frac{(-1)^\nu}{\nu!} \frac{1}{z+\nu} \qquad (n > 2)$$

has all its real roots lying one in each of the $2n - 3$ intervals:

$$-2m - 1 - \frac{6}{(2m)!} \leqq z \leqq -2m - 1 - \frac{1}{(2m)!} \qquad (m = 2, 3, \cdots, n)$$

$$-2m - 2 + \frac{1}{(2m+1)!} \leqq z \leqq -2m - 2 + \frac{6}{(2m+1)!} \qquad (m = 2, 3, \cdots, n-1),$$

and hence that $P_n(z)$ has four complex roots and from this proceeding to $P(z)$ itself by applying Rouché's theorem. The method of computation consists in locating the complex zeros of $P_{10}(z)$ to three decimal places and then, using these as initial values, finding the roots of $P(z) = 0$ by a direct application of Newton's method of approximation.

Starting with the mean values of the above-mentioned intervals for $n = 10$, and applying Newton's method of approximation to

$$P_{10}(z) = \sum_{\nu=0}^{21} \frac{(-1)^\nu}{\nu!} \frac{1}{z+\nu} \tag{1}$$

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† L. Bourguet, Sur la théorie des intégrales Euleriennes, Comptes rendus, vol. 96 (1883), pp. 1487–1490.

‡ T. H. Gronwall, Sur les zéros des fonctions $P(z)$ et $Q(z)$ associées à la fonction gamma, Annales de l'Ecole Normale, Ser. III, vol. 33 (1916), pp. 381–393.

we find that the real roots are

$$p_5 = -5.11907; \qquad p_6 = -5.97158; \qquad p_7 = -7.00432;$$

$$p_8 = -7.99940; \qquad p_9 = -9.00008; \qquad p_{10} = -9.99999;$$

$$p_m = -m \qquad (m = 11, 12, \cdots 21),$$

where, as in the rest of this computation, quantities which do not affect the fifth decimal place are neglected. Hence $P_{10}(z)$ which may be written as

$$\frac{a_0 z^{21} + a_1 z^{20} + \cdots + a_{20} z + a_{21}}{z(z+1)(z+2)\cdots(z+21)}$$

is equal to

$$\frac{b_0 z^{10} + b_1 z^9 + \cdots + b_9 z + b_{10}}{z(z+1)(z+2)\cdots(z+10)} \tag{2}$$

to the desired degree of accuracy.

If $P_{10}(z)$ be expanded in powers of $1/z$, the coefficient of $1/z^r$ is

$$\sum_{\nu=1}^{21} \frac{(-1)^{\nu+r}\nu^r}{\nu!},$$

and since this is sensibly equal to

$$\sum_{\nu=1}^{\infty} \frac{(-1)^{\nu+r}\nu^r}{\nu!} \qquad (r < 10),$$

$$\frac{1}{e}\left(\frac{1}{z} + \frac{1}{z^2} - \frac{1}{z^4} + \frac{1}{z^5} + \cdots\right) \tag{3}$$

is obtained as a second approximate form for $P_{10}(z)$. A comparison of (3) with (2) gives $b_0 = 1/e$, $b_1 = 56/e$, $b_2 = 1375/e$.

The expansion of $P_{10}(z)$ in the form

$$\frac{1}{z} + A_1 + A_2 z + A_3 z^2 + \cdots \tag{4}$$

(where $A_1 = \Sigma_{\nu=1}^{21}[(-1)^\nu/\nu!\nu] = .7965996$) when compared with (2) determines the values $b_9 = 7{,}737{,}939$, $b_{10} = 3{,}628{,}800$.

The numerator of (2) (with the factor $1/e$ removed) is therefore equal to

$$z^{10} + 56z^9 + 1375z^8 + \cdots + 21{,}033{,}900z + 9{,}864{,}102. \tag{5}$$

This expression, set equal to zero, has for roots the four complex roots of $P_{10}(z) = 0$, together with p_5, p_6, $\cdots$ p_{10} found above. The product $\Pi_{m=5}^{10}(z - p_m)$ is found to be

$$z^6 + 45.0944z^5 + 355.917z^4 + \cdots + 129{,}749z + 154{,}152. \tag{6}$$

Dividing (5) by (6), we obtain the equation

$$z^4 + 10.906z^3 + 44.425z^2 + 82.589z + 63.989 = 0. \tag{7}$$

The determination of the fourth and fifth terms of (7) from the last two terms of (5) and (6) is preferable to the use of the first five terms of (5) and (6), as it involves fewer operations and consequently gives these terms more accurately than the alternative method.

The roots of (7), obtained by the use of Ferrari's formulæ, are, to three decimal places:

$$p_1 = -1.726 + 1.238i, \qquad p_2 = -1.726 - 1.238i,$$

$$p_3 = -3.727 + .544i, \qquad p_4 = -3.727 - .544i.$$

Starting with these values, and applying Newton's method of approximation to $P(z)$ itself, we find that the four complex roots of $P(z)$ are

$$z_1 = -1.7262976 + 1.2380921i, \qquad z_2 = -1.7262976 - 1.2380921i,$$

$$z_3 = -3.7264730 + .5406746i, \qquad z_4 = -3.7264730 - .5406746i.$$

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